

LINES ON K3 QUARTIC SURFACES IN CHARACTERISTIC 3

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ABSTRACT. We prove that a K3 quartic surface X defined over a field of characteristic 3 can contain at most 112 lines. If X contains 112 lines, then it is projectively equivalent to the Fermat quartic surface; otherwise, X contains at most 67 lines. If X contains a star, then it has 112 lines or at most 58. We provide explicit equations of three 1-dimensional families of smooth quartic surfaces with 58 lines.

1. INTRODUCTION

In the last years the enumerative geometry of straight lines on quartic surfaces in \mathbb{P}^3 has been studied by many authors.

Unlike smooth cubic surfaces, which always contain 27 lines, one can prove by a standard dimension count that a general quartic surface does not contain any line at all. One is therefore led to study the maximal number of lines that a quartic surface can contain.

Historically, the main focus has been on smooth complex quartic surfaces, which are well-known examples of algebraic K3 surfaces. In 1882 F. Schur [13] discovered a surface with 64 lines that now carries his name, given by the following equation

$$x^4 - xy^3 = w^4 - wz^3.$$

The fact that 64 is indeed the highest number of lines that a smooth complex quartic surface can contain was proven by B. Segre in 1943 [15]. Around 70 years later, though, S. Rams and M. Schütt [10] discovered a flaw in Segre's argument and fixed his proof, extending it to smooth quartic surfaces defined over a field of characteristic different from 2 and 3.

At the same time, A. Degtyarev, I. Itenberg and A. S. Sertöz [6] – spurred by a remark of W. Barth [1] – were tackling the same problem using the theory of K3 surfaces and Nikulin's theory of discriminant forms. Their work resulted in a complete classification of large configurations of lines on smooth complex quartic surfaces up to projective equivalence. They also proved that Schur's quartic is the only one containing 64 lines (this holds not only over \mathbb{C} , but also over any field of characteristic different from 2 and 3, see [5]).

Segre–Rams–Schütt's theorem was generalized by the author [19] to K3 quartic surfaces – i.e., quartic surfaces containing at worst rational double points as singularities – defined over a field of characteristic different from 2 and 3. The bound of 64 lines is still not known to be sharp for K3 quartic

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surfaces. Degtyarev (private communication) has proven the existence of a complex K3 quartic surface with 52 lines and 2 singular points of type \mathbf{A}_1 , whose equation is not known.

Non-K3 complex quartic surfaces have been studied by González Alonso and Rams [7]. If they are not ruled by lines, they can contain at most 48 lines. They also conjecture that the actual bound is 31.

Given a line ℓ on a K3 quartic surface X , the pencil of planes in \mathbb{P}^3 containing ℓ induces a genus 1 fibration on the minimal desingularization of X , i.e. an elliptic or a quasi-elliptic fibration: the latter case can only appear in characteristic 2 and 3, by a theorem of J. Tate [17]. Heuristically, quasi-elliptic fibrations can carry a higher number of fiber components than elliptic fibrations, thus allowing for a higher number of lines on X .

In characteristic 2, X can contain at most 68 lines [18]. Moreover, if X contains 68 lines, then X is projectively equivalent to a member of a certain 1-dimensional family discovered by Rams and Schütt.

On the other hand, the genus 1 fibration induced by a line ℓ can be quasi-elliptic only if ℓ contains a singular point. Thus, if X is smooth, then the fibration is always elliptic. This fact was proven by Rams and Schütt in [11], where they also exhibited a smooth surface with 60 lines. Degtyarev [5] subsequently proved that 60 is indeed the maximum possible.

In this paper we are concerned with K3 quartic surfaces defined over a field of characteristic 3. The Fermat quartic surface defined by

$$x^4 + y^4 + w^4 + z^4 = 0$$

contains 112 lines (see, for instance, [2]). Our main result is the following theorem.

Theorem 1.1. *If X is a K3 quartic surface defined over a field of characteristic 3, then X contains 112 lines or at most 67. Moreover, if X contains 112 lines, then X is projectively equivalent to the Fermat quartic surface.*

This theorem extends a similar result by Rams and Schütt [9] on smooth quartic surfaces. The bound of 67 lines is not known to be sharp.

Degtyarev [5] has proven a bound of 60 lines for smooth quartic surfaces not projectively equivalent to the Fermat quartic surface, which is not known to be sharp either. The number 60, though, could only be attained by a non-supersingular surface. He also showed that smooth supersingular quartic surfaces have 112 lines or at most 58, and that there are three different possible configurations of 58 lines. In Examples 6.1, 6.2 and 6.3 we provide explicit equations of three 1-dimensional families of smooth quartic surfaces containing these configurations. The first and second families had already been found also by Degtyarev, while the third one is new, to our knowledge.

The three families were found while examining the proof of Proposition 5.13 (for the third one, also taking advantage of Degtyarev's explicit configuration of lines, which was communicated to us by email), which states that if X contains a star, i.e., four (coplanar) lines intersecting at a smooth point, then X contains at most 58 lines. This provides evidence for the following conjecture.

Conjecture 1.2. *A K3 quartic surfaces defined over a field of characteristic 3 contains 112 lines or at most 58.*

The bound of 67 lines could possibly be improved by distinguishing between supersingular and non-supersingular K3 quartic surfaces. For the former, a deep lattice-theoretical analysis seems necessary. For the latter, our geometrical approach could probably lead to a bound of 64 lines.

The paper is structured as follows.

- §2: We set the notation and present some general facts on K3 quartic surfaces defined over a field of any characteristic.
- §3: We study elliptic lines. Not all arguments in characteristic 0 carry over to characteristic 3, mainly because they were concerned with 3-torsion sections, which are now not so well behaved.
- §4: We examine a new phenomenon that does not appear in characteristic 0, namely quasi-elliptic lines: these are very important because they exhibit particularly high valencies.
- §5: We carry out the proof of Theorem 1.1. We prove a better estimate – 58 lines – under the hypothesis that X contains a star and is not projectively equivalent to the Fermat quartic surface, see Proposition 5.13.
- §6: We discuss some examples of K3 quartic surfaces with many lines. In particular, we present three 1-dimensional families of smooth surfaces with 58 lines and a surface with 8 singular points and 48 lines.

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2. K3 QUARTIC SURFACES

Given a field \mathbb{K} of characteristic $p \geq 0$, a *K3 quartic surface* is a surface in \mathbb{P}^3 of degree 4 with at most rational double points as singularities. In this section we fix our notation while recalling some general facts about K3 quartic surfaces. For more details the reader is referred to [18, §2].

Henceforth, X will denote a fixed K3 quartic surface, $\text{Sing}(X)$ the set of (isolated) singular points on X and $\rho : Z \rightarrow X$ the minimal desingularization of X .

2.1. Lines and singular points. From now on, ℓ will denote a line lying on a K3 quartic surface X with minimal desingularization $\rho : Z \rightarrow X$. Any divisor in the complete linear system defined by $H := \rho^*(\mathcal{O}_X(1))$ will be called a *hyperplane divisor* (and often denoted by H , too). The strict transform of ℓ will be denoted by L .

The pencil of planes $\{\Pi_t\}_{t \in \mathbb{P}^1}$ containing the line ℓ induces a genus 1 fibration $\pi : Z \rightarrow \mathbb{P}^1$. A line ℓ is said to be *elliptic* (respectively *quasi-elliptic*) if it induces an elliptic (respectively *quasi-elliptic*) fibration. A fiber F_t of π ($t \in \mathbb{P}^1$) is the pullback through ρ of the residual cubic E_t contained in Π_t .

We denote the restriction of π to L again by π . If the morphism $\pi : L \rightarrow \mathbb{P}^1$ is constant, we say that L has degree 0; otherwise, the *degree* of ℓ is the degree of the morphism $\pi : L \rightarrow \mathbb{P}^1$. A line ℓ is said to be *separable* (respectively *inseparable*) if the induced morphism $\pi : L \rightarrow \mathbb{P}^1$ is separable (respectively inseparable).

The *singularity* of a line ℓ is the number of singular points of X lying on ℓ .

Given a separable line ℓ , we will say that a point P on ℓ is a point of ramification n_m if the corresponding point on L has ramification index n and $\text{length}(\Omega_{L/\mathbb{P}^1}) = m$. We recall that if $\text{char } \mathbb{K}$ does not divide n , then $m = n - 1$ and can be omitted, whereas if $\text{char } \mathbb{K}$ divides n , then $m \geq n$.

For the proof of the following lemma, see [19, Lemma 2.7].

Lemma 2.1. *If P is a singular point on a K3 quartic surface X , then there are at most 8 lines lying on X and passing through P .*

2.2. Valency. Given a K3 quartic surface X , we will denote by $\Phi(X)$ the number of lines lying on X .

Building on an idea of Segre [15], we will usually be interested in finding a *completely reducible plane*, i.e., a plane Π such that the intersection $X \cap \Pi$ splits into the highest possible number of irreducible components, namely four lines ℓ_1, \dots, ℓ_4 (not necessarily distinct). If a line ℓ' not lying on Π meets two or more distinct lines ℓ_i , then their point of intersection must be a singular point of the surface. It follows that $\Phi(X)$ is bounded by

$$(1) \quad \begin{aligned} \Phi(X) &\leq \#\{\text{lines in } \Pi\} \\ &\quad + \#\{\text{lines not in } \Pi \text{ going through } \Pi \cap \text{Sing}(X)\} \\ &\quad + \sum_{i=1}^4 \#\{\text{lines not in } \Pi \text{ meeting } \ell_i \text{ in a smooth point}\}. \end{aligned}$$

It will then be a matter of finding a bound for the second and third contribution. The former will be usually dealt with using Lemma 2.1. As for the latter, it is natural to introduce the following definition.

Definition 2.2. The *valency* of ℓ , denoted by $v(\ell)$, is the number of lines on X distinct from ℓ which intersect ℓ in smooth points.

Most of the time we will express the latter contribution in terms of $v(\ell_i)$, and much of the work will be dedicated to finding a bound for these quantities. Of course, not all K3 quartic surfaces admit a completely reducible plane, in which case we will turn to other techniques, such as the ones presented in [19, §5].

Definition 2.3. A *3-fiber* is a fiber whose residual cubic splits into three lines, whereas a *1-fiber* is a fiber whose residual cubic splits into a line and an irreducible conic. A line ℓ is said to be *of type* (p, q) , $p, q \geq 0$, if in its fibration there are p fibers of the former kind and q fibers of the latter kind.

Definition 2.4. The (local) valency of a fiber F , denoted by $v_\ell(F)$, is the number of lines distinct from ℓ contained in the plane corresponding to F that meet ℓ in a smooth point. When it is clear from the context, we will simply write $v(F)$.

Obviously,

$$(2) \quad v(\ell) = \sum_{t \in \mathbb{P}^1} v_\ell(F_t),$$

and the sum is actually a finite sum. It follows from this formula that if ℓ has type (p, q) , then

$$(3) \quad v(\ell) \leq dp + q.$$

2.3. Lines of the first and second kind. Let x_0, x_1, x_2, x_3 be the coordinates of \mathbb{P}^3 . Up to projective equivalence, we can suppose that the line ℓ is given by the vanishing of x_0 and x_1 , so that the quartic X is defined by

$$(4) \quad X : \sum_{i_0+i_1+i_2+i_3=4} a_{i_0 i_1 i_2 i_3} x_0^{i_0} x_1^{i_1} x_2^{i_2} x_3^{i_3} = 0, \quad a_{i_0 i_1 0 0} = 0 \text{ for all } i_0, i_1,$$

where i_0, \dots, i_4 are non-negative integers.

Remark 2.5. We will usually parametrize the planes containing ℓ by Π_t : $x_0 = tx_1$, $t \in \mathbb{P}^1$, where of course $t = \infty$ denotes the plane $x_1 = 0$. Two equations which define the *residual cubic* E_t contained in Π_t are the equation of Π_t itself and the equation $g \in \mathbb{K}[t][x_1, x_2, x_3]_{(3)}$ obtained by substituting x_0 with tx_1 in (4) and factoring out x_1 . An explicit computation shows that the intersection of ℓ with E_t is given by the points $[0 : 0 : x_2 : x_3]$ satisfying

$$(5) \quad g_t(0, x_2, x_3) = t\alpha(x_2, x_3) + \beta(x_2, x_3) = 0.$$

Given a line ℓ of positive degree, a crucial technique to find bounds for $v(\ell)$ is to count the points of intersection of the residual cubics E_t and ℓ which are inflection points for E_t . By *inflection point* we mean here a point which is also a zero of the hessian of the cubic. In fact, if a residual cubic E_t contains a line as a component, all the points of the line will be inflection points of E_t .

Writing out the equation of a cubic in \mathbb{P}^2 explicitly and computing the determinant of its hessian matrix, one can also check the following lemma.

Lemma 2.6. *Let E be a reducible cubic in \mathbb{P}^2 that is the union of an irreducible conic and a line ℓ' . Then, the locus of inflection points of E is exactly ℓ' .*

Supposing that the surface X is defined as in equation (4), the hessian of the equation g defining the residual cubic E_t (see Remark 2.5) restricted on the line ℓ is given by

$$(6) \quad h := \det \left(\frac{\partial^2 g}{\partial x_i \partial x_j} \right)_{1 \leq i, j \leq 3} \Big|_{x_1=0} \in \mathbb{K}[t][x_2, x_3]_{(3)},$$

which is a polynomial of degree 5 in t , with forms of degree 3 in (x_2, x_3) as coefficients (if $\text{char } \mathbb{K} = 2$, this definition has to be slightly modified, see [9]).

We want now to find the number of lines intersecting ℓ by studying the common solutions of (5) and (6). It is convenient to extend Segre's nomenclature [15].

Definition 2.7. The resultant $R(\ell)$ with respect to the variable t of the polynomials (5) and (6) is called the *resultant* of the line ℓ .

Definition 2.8. We say that a line ℓ of positive degree is a line of the *second kind* if its resultant is identically equal to zero. Otherwise, we say that ℓ is a line of the *first kind*.

A root $[\bar{x}_2 : \bar{x}_3]$ of $R(\ell)$ corresponds to a point $P = [0 : 0 : \bar{x}_2 : \bar{x}_3]$ on ℓ ; if P is a smooth surface point, then it is an inflection point for the residual cubic passing through it.

Proposition 2.9. *If ℓ is a line of the first kind, then $v(\ell) \leq 3 + 5d$.*

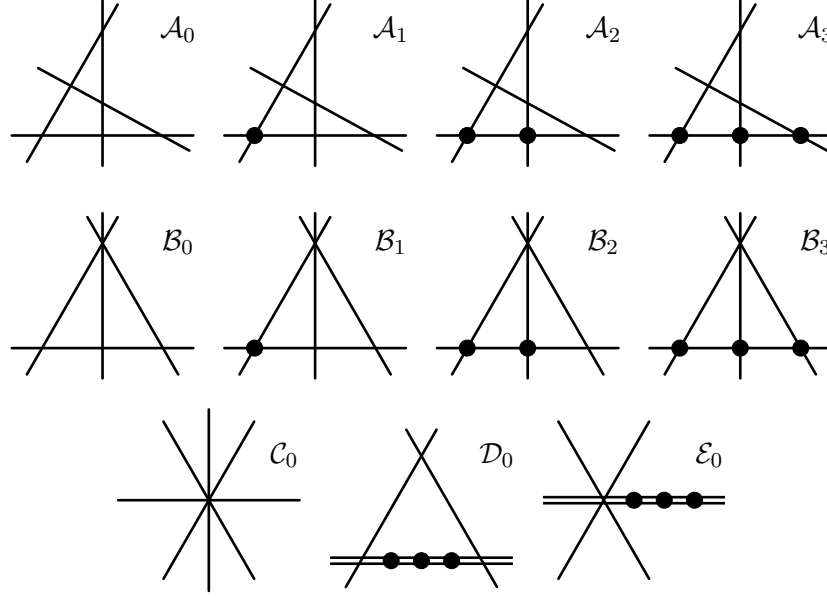


FIGURE 1. Possible configurations of lines on a plane with a triangle. Singular points are marked with a bullet. In configurations \mathcal{D}_0 and \mathcal{E}_0 the singular points might coincide.

Proof. See [18, Proposition 2.18]. \square

2.4. Triangle free surfaces. We follow here the nomenclature of [19, §5]. In particular, the *line graph* of a K3 quartic surface X is the dual graph of the strict transforms of its lines on Z . The line graph $\Gamma = \Gamma(X)$ of a K3 quartic surface X is a graph without loops or multiple edges. By definition, the number of its vertices is equal to $\Phi(X)$.

A Dynkin diagram (resp. extended Dynkin diagram) is also called an *elliptic graph* (resp. *parabolic graph*).

A K3 quartic surface X is called *triangle free* if its line graph contains no triangles, i.e., cycles of length 3. In other words, a K3 quartic surface X is triangle free if there are no triples of lines on X intersecting pairwise in smooth points.

Lemma 2.10. *If ℓ is an elliptic line on a triangle free K3 quartic surface, then $v(\ell) \leq 12$.*

Lemma 2.11. *If three lines on X form a triangle, then they are contained in plane Π such that the intersection of Π and X has one of the configurations pictured in Figure 1.*

For a proof, see [18, Lemmas 2.22 and 2.23]. The nomenclature for completely reducible planes introduced in Figure 1 will be used also in the next sections.

3. ELLIPTIC LINES

From now on we assume that the ground field \mathbb{K} is algebraically closed and has characteristic 3.

In this section we study elliptic lines, especially separable elliptic lines. Inseparable lines (both elliptic and quasi-elliptic) will be analyzed in §4. The results of this section are summarized in Table 1.

TABLE 1. Known bounds for the valency of a separable elliptic line according to its kind, degree and singularity. Sharp bounds are marked with an asterisk.

kind	degree	singularity	valency
first kind	3	0	$\leq 18^*$
	2	1	≤ 13
	1	2 or 1	≤ 8
second kind	3	0	$\leq 21^*$
	2	1	$\leq 14^*$
	1	2	≤ 9
	1	1	≤ 11
–	0	3, 2 or 1	$\leq 2^*$

Lines of degree 0 have valency less than or equal to 2 by [19, Lemma 2.6], whereas lines of the first kind have already been treated in and Proposition 2.9, respectively. In this section we will therefore concentrate on lines of the second kind.

The following two lemmas will also be useful in the study of quasi-elliptic lines.

Lemma 3.1. *Let ℓ be a separable line of the second kind and $P \in \ell$ a smooth point of ramification 2. Then, either the corresponding fiber is of type II with a cusp in P , or the corresponding residual cubic splits into a double line plus a simple line.*

Proof. Note that only lines of degree 3 and 2 can have a point P of ramification 2. We choose coordinates so that P is given by $[0 : 0 : 0 : 1]$. This means that

$$a_{0103} = 0 \quad \text{and} \quad a_{0112} = 0.$$

Since P is of ramification index 2 and it is non-singular, by rescaling variables we can normalize

$$a_{0121} = 1 \quad \text{and} \quad a_{1003} = 1.$$

Since ℓ is of the second kind, the following relations must be satisfied:

$$a_{0202} = 0, \quad a_{0301} = a_{0211}^2 \quad \text{and} \quad a_{0310} = a_{0211}a_{0220}.$$

This means that the residual cubic in $x_0 = 0$ corresponding to P is given by

$$(a_{0211}x_1 - x_2)^2 x_3 + f_3(x_1, x_2),$$

where f_3 is a form of degree 3. Either this cubic is irreducible and gives rise to a fiber of type II, or the polynomial $m = a_{0211}x_1 - x_2$ divides f_3 ; in the latter case it is immediate to compute that also m^2 divides f_3 . \square

Lemma 3.2. *Let ℓ be a separable line of the second kind and $P \in \ell$ a point of ramification 3_4 . Then, either the corresponding fiber is of type II with a cusp in P , or the corresponding residual cubic splits into three concurrent lines (not necessarily distinct).*

Proof. Note that ℓ has necessarily degree 3. We choose coordinates so that P is given by $[0 : 0 : 0 : 1]$ and the fiber corresponds to the plane $\Pi_0 : x_0 = 0$. This means that

$$a_{0103} = 0, \quad a_{0112} = 0 \quad \text{and} \quad a_{0121} = 0.$$

A calculation with local parameters shows that $\text{length}(\Omega_{L/\mathbb{P}^1}) = 4$ if and only if $a_{1012} = 0$. Moreover, the following three coefficients must be different from 0: a_{0130} , a_{1003} and a_{1021} ; the first two because otherwise there would be singular points on ℓ (implying that the degree of ℓ is less than 3), the third because otherwise ℓ would be inseparable. We can normalize them to 1, rescaling coordinates. Two necessary conditions for the line ℓ to be of the second kind are

$$a_{0202} = 0 \quad \text{and} \quad a_{0211} = 0.$$

Hence, the residual cubic in Π_0 is given by

$$a_{0301}x_1^2x_3 + x_2^3 + x_1f_2(x_1, x_2).$$

It is then clear that either the fiber is irreducible and has a cusp in P ($a_{0301} \neq 0$), or it splits into three concurrent lines ($a_{0301} = 0$). \square

Remark 3.3. Suppose that ℓ is an elliptic line of degree 3 with fibration $\pi : Z \rightarrow \mathbb{P}^1$. Denoting by δ_t the wild ramification index of the fiber F_t , $t \in \mathbb{P}^1$ (see, for instance, [4]), observe that

$$v_\ell(F_t) \leq e(F_t) \leq e(F_t) + \delta_t,$$

since a reducible fiber has $e(F_t) \geq 2$ and a fiber with at least three components has $e(F_t) \geq 3$. From equations (2) and the Euler–Poincaré characteristic formula (see, for instance, [4, Proposition 5.16]) we infer that

$$(7) \quad v(\ell) = \sum_{t \in \mathbb{P}^1} v_\ell(F_t) \leq \sum_{t \in \mathbb{P}^1} (e(F_t) + \delta_t) = e(Z) = 24.$$

The only fiber type whose Euler–Poincaré characteristic is equal to its contribution to the valency of ℓ is type I_3 . Hence, if for any subset $S \subset \mathbb{P}^1$ one has

$$\sum_{s \in S} e(F_s) = \sum_{s \in S} v_\ell(F_s) = N,$$

then all fibers F_s must be of type I_3 and, in particular, N must be divisible by 3.

An application of the Riemann–Hurwitz formula yields the following lemma.

Lemma 3.4. *If ℓ is a separable line of degree 3, then ℓ has ramification 2_1^4 , 2_13_3 or 3_4 .*

Proposition 3.5. *Let ℓ be a separable elliptic line of the second kind of degree 3. Then, the valency of ℓ is bounded according to the following table, where sharp bounds are marked with an asterisk:*

<i>ramification</i>	<i>valency</i>
2_1^4	≤ 12
$2_1 3_3$	$\leq 21^*$
3_4	$\leq 21^*$

Proof. Suppose first that ℓ has a point P of ramification index 2. According to Lemma 3.1, the corresponding fiber F_P is either of type II, so that $e(F_P) + \delta_P \geq 2 + 1 = 3$ (type II has wild ramification – see [14, Proposition 16]) and $v(F_P) = 0$, or it contains a double component, so that $e(F_P) \geq 6$ and $v(F_P) = 2$; in any case, the difference $e(F_P) + \delta_P - v(F_P)$ is always at least 3. Therefore, if there are 4 points of ramification 2, then by formula (7) $v(\ell)$ is not greater than $24 - 4 \cdot 3 = 12$, while if there is just one, $v(\ell)$ is not greater than $24 - 3 = 21$.

Suppose now that ℓ has no point of ramification index 2, i.e., ℓ has ramification 3_4 . If the ramified fiber F_0 is of type II, then there can be at most $24 - 3 = 21$ lines meeting ℓ . If F_0 splits into three concurrent lines, then $e(F_0) + \delta_0 \geq 5$ (type IV has wild ramification, too), which means that the contribution to $v(\ell)$ of the other fibers is not greater than $24 - 5 = 19$. Nonetheless, by Remark 3.3 this contribution cannot be exactly 19, since 19 is not divisible by 3; hence, again, we can have at most 18 lines meeting ℓ . \square

Example 3.6. The following surface contains a separable line ($x_0 = x_1 = 0$) of ramification $2_1 3_3$ with valency 21:

$$x_0^4 + x_0^2 x_1 x_2 - x_1^3 x_2 + x_0 x_1 x_2^2 + x_1 x_2^3 + x_0^2 x_1 x_3 + x_1^2 x_3^2 + x_0 x_2 x_3^2 + x_0 x_3^3 = 0.$$

Example 3.7. The following surface contains a separable line $x_0 = x_1 = 0$ of ramification 3_4 with valency 21:

$$\begin{aligned} & i x_0^3 x_1 + i x_1^3 x_2 + i x_1 x_2^3 - i x_0^3 x_3 + i x_0 x_1 x_2 x_3 + i x_0 x_3^3 \\ & = x_0^2 x_1 x_2 + x_1^2 x_2^2 + x_0 x_2^2 x_3 - x_0^2 x_3^2, \end{aligned}$$

where i is a square root of -1 .

Proposition 3.8. *If ℓ is an elliptic line of degree 2, then, $v(\ell) \leq 14$.*

Proof. By Proposition 2.9, we can assume that ℓ is of the second kind. Since ℓ has degree 2, it must have singularity 1: let P be the singular point on ℓ . The morphism $\pi : L \rightarrow \mathbb{P}^1$, being of degree 2, is separable and has two points of ramification index 2. At least one of the point of ramification must be different from P : let us call it Q . By Lemma 3.1 either the fiber corresponding to Q is of type II or the residual cubic splits into a double line and a simple line.

- Suppose the fiber F_Q is of type II. If ℓ is of type (p, q) , then $3p + 2q \leq 24 - 3 = 21$. Applying formula (3), we have

$$v(\ell) \leq 2p + q = 14.$$

- If the residual cubic corresponding F_Q splits into a double line and a simple line, then it contributes 1 to the valency and at least 6 to the Euler–Poincaré characteristic. Applying formula (3) again yields $v(\ell) \leq 13$. \square

Example 3.9. The following surface contains an elliptic line $\ell : x_0 = x_1 = 0$ of degree 2 with valency 14, thus attaining the bound in Proposition 3.8. The surface contains one point $P = [0 : 0 : 0 : 1]$ of type \mathbf{A}_1 . The line ℓ has 7 fibers of type \mathbf{I}_3 and one ramified fiber of type \mathbf{II} . The other ramified fiber corresponds to the plane $x_0 = 0$ and is smooth.

$$x_0^4 + x_0^2 x_1 x_2 - x_1^3 x_2 + x_0 x_1 x_2^2 + x_1 x_2^3 + x_1^2 x_3^2 + x_0 x_2 x_3^2 = 0.$$

Proposition 3.10. *Let ℓ be an elliptic line of degree 1. Then, $v(\ell) \leq 9$ if ℓ has singularity 2, and $v(\ell) \leq 11$ if ℓ has singularity 1.*

Proof. The proof can be carried over word by word from the characteristic 0 case (see [19, Propositions 2.13 and 2.14]). \square

4. QUASI-ELLIPTIC LINES

The phenomenon of quasi-elliptic lines is arguably the main difference with the characteristic 0 case. We will first recall some general facts about quasi-elliptic fibrations in characteristic 3 (see [3, 4, 12]).

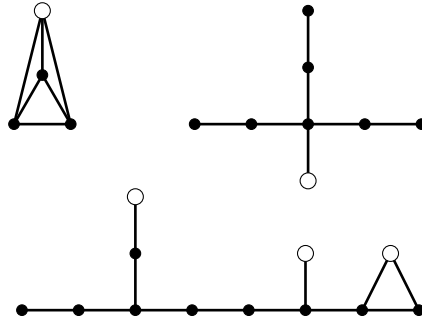
In characteristic 3 only the following fiber types can arise in a quasi-elliptic fibration (for simplicity, we call them *quasi-elliptic fibers*):

$$\mathbf{II}, \mathbf{IV}, \mathbf{IV}^*, \mathbf{II}^*.$$

We will denote by iv , iv^* and ii^* the number of fibers of type \mathbf{IV} , \mathbf{IV}^* and \mathbf{II}^* , respectively. On a K3 surface, the Euler–Poincaré characteristic formula takes the following form:

$$(8) \quad iv + 3iv^* + 4ii^* = 10.$$

It is well known (see [3, 12]) that the cuspidal curve of a quasi-elliptic fibration is a smooth curve K such that $K \cdot F = 3$. The restriction of the fibration to K is an inseparable morphism of degree 3. The cuspidal curve meets a degenerate fiber in the following ways (multiple empty dots represent different possibilities):



We note that

- K intersects a fiber of type \mathbf{IV} at the intersection point of the three components.
- The way K intersects F is uniquely determined unless F is of type \mathbf{II}^* .

4.1. Quasi-elliptic lines of degree 3. Quasi-elliptic lines of degree 3 play a crucial role, mainly because it is the only case where the strict transform L of the line itself can serve as the cuspidal curve.

Definition 4.1. A line ℓ is said to be *cuspidal* if it is quasi-elliptic and the cuspidal curve K of the induced fibration coincides with the strict transform L .

Table 2 summarizes the known bounds for the valency of a quasi-elliptic line, which will be proven in this section.

TABLE 2. Known bounds for the valency of a quasi-elliptic line. Sharp bounds are marked with an asterisk.

degree	valency
3	cuspidal $\leq 30^*$
	not cuspidal $\leq 21^*$
2	$\leq 14^*$
1	≤ 10
0	≤ 2

Since the restriction of the fibration on K is an inseparable morphism $K \rightarrow \mathbb{P}^1$, a cuspidal line is necessarily inseparable. The following lemma gives a bound on the valency for inseparable lines which are not cuspidal.

Lemma 4.2. *If ℓ is an inseparable line and $v(\ell) > 12$, then ℓ is cuspidal.*

Proof. Up to coordinate change, we can suppose that the residual cubic contained in $x_0 = tx_1$ intersects the line $\ell : x_0 = x_1 = 0$ in $[0 : 0 : 0 : 1]$ for $t = 0$ and in $[0 : 0 : 1 : 0]$ for $t = \infty$. This means that the following coefficients vanish:

$$a_{0103}, a_{0112}, a_{0121}; a_{1012}, a_{1021}, a_{1030}.$$

Moreover, a_{1003} and a_{0130} must be different from 0, and can be normalized to 1 and -1 , respectively, by rescaling coordinates. Up to a Frobenius change of parameter $t = s^3$, we can explicitly write the intersection point P_s of the residual cubic with ℓ , which is given by

$$P_s = [0 : 0 : s : 1].$$

If a residual cubic E_s is reducible, then all components must pass through P_s ; in particular, P_s must be a singular point of E_s . One can see explicitly that P_s is a singular point of E_s if and only if s is a root of the following degree 8 polynomial:

$$(9) \quad \begin{aligned} \varphi(s) := & a_{2020}s^8 + a_{2011}s^7 + a_{2002}s^6 + a_{1120}s^5 \\ & + a_{1111}s^4 + a_{1102}s^3 + a_{0220}s^2 + a_{0211}s + a_{0202}. \end{aligned}$$

Furthermore, it can be checked by a local computation that if E_s splits into three (not necessarily distinct) lines, then s is a double root of φ . This implies that the valency of ℓ is not greater than $3 \cdot 8/2 = 12$, unless the polynomial φ vanishes identically, but $\varphi \equiv 0$ implies that all points P_s are singular for E_s , i.e., the line ℓ is cuspidal. \square

Corollary 4.3. *If $\ell \subset X$ is cuspidal, then X is projectively equivalent to a member of the family \mathcal{C} defined by*

$$\mathcal{C} := x_0x_3^3 - x_1x_2^3 + x_2q_3(x_0, x_1) + x_3q_3'(x_0, x_1) + q_4(x_0, x_1),$$

where q_3 , q_3' and q_4 are forms of degree 3, 3 and 4, respectively.

Proof. The family can be found imposing that φ vanishes identically. \square

Corollary 4.4. *If ℓ is cuspidal, then a residual cubic corresponding to a reducible fiber of ℓ is either the union of three distinct concurrent lines, or a triple line.*

Proof. The intersection of a residual cubic with ℓ is always one single point.

A residual cubic of ℓ cannot be the union of a line and an irreducible conic, because the line and the conic would result in a fiber of type I_n (because the conic has to be tangent to ℓ), which is not quasi-elliptic.

Therefore, a residual cubic relative to a degenerate fiber must split into three (not necessarily distinct) lines. If at least two of them coincide, the plane on which they lie contains at least a singular point P of the surface (which is not on ℓ , since ℓ has degree 3). An explicit inspection of this configuration in the family \mathcal{C} (for instance, supposing up to change of coordinates that P is given by $[0 : 1 : 0 : 0]$) shows that the residual cubic degenerates to a triple line. \square

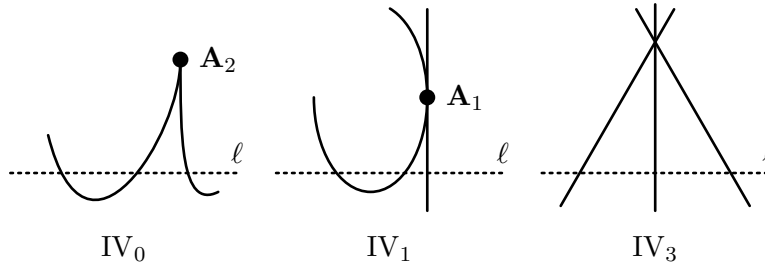
Lemma 4.5. *If ℓ is a quasi-elliptic line of degree 3, then $v(\ell) \leq 30$.*

Proof. The fibration induced by the line ℓ has at most 10 reducible fibers, each of which can contribute at most 3 to its valency. \square

Remark 4.6. The bound of Lemma 4.5 is sharp. As soon as a K3 quartic surface X is smooth, the valency of a quasi-elliptic line of degree 3 on X is automatically 30, because the fibration induced by ℓ can only have 10 reducible fibers of type IV, whose residual cubics are the union of three concurrent lines. Notably, this happens for all 112 lines on the Fermat surface.

We will now prove that a quasi-elliptic line needs to be cuspidal in order to have valency greater than 21.

Lemma 4.7. *Let ℓ be any line of degree 3. A fiber of type IV must have one of the following residual cubics, with the only restriction that ℓ cannot pass through a singular point:*



Proof. A fiber of type IV contains three simple components; hence, the corresponding residual cubic can also have only simple components. Since it cannot contain cycles, it must be one of the following, as in the picture:

- a cusp;
- a conic and a line meeting tangentially in one point;
- three distinct lines meeting in one point.

The remaining components must come from the resolution of the singular points on the surface. The types of the singular points can be immediately deduced from the respective Dynkin diagrams. \square

Lemma 4.8. *Let ℓ be any line of degree 3. A fiber of type IV^* must have one of the residual cubics pictured in Figure 2, with the only restriction that ℓ cannot pass through a singular point.*

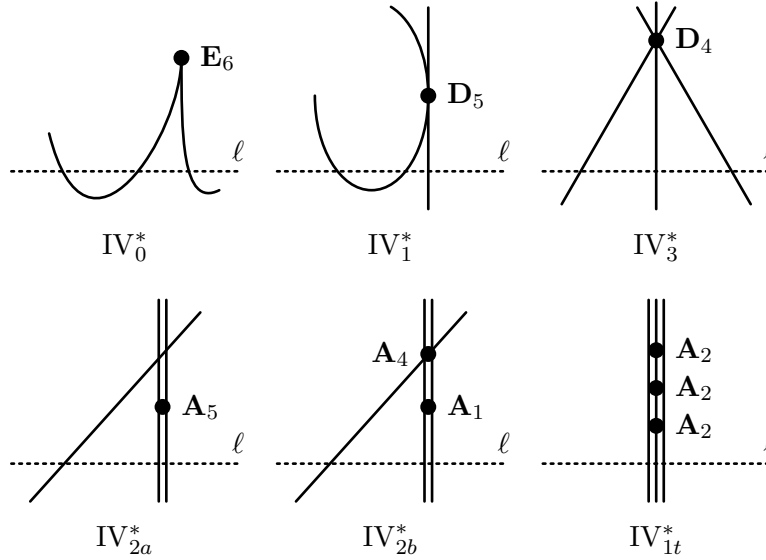


FIGURE 2. Possible residual cubics corresponding to a fiber of type IV^* .

Proof. Besides the residual cubics with only simple components described in the previous lemma, we can also have multiple components, namely

- a double line and a simple line;
- a triple line.

In the former case, the strict transforms of the lines can intersect (if their intersection point is smooth) or not (if their intersection point is singular), giving rise to two different configurations, which we distinguish by the letters a and b . In the latter case, there is no ambiguity, since a fiber of type IV^* contains only one triple component. \square

From now on we will denote by iv_0, iv_1, \dots the number of fibers of type IV_0, IV_1 , and so on. Note that the subscript indicates the local valency of the fiber.

As a last ingredient, we need to find a bound for the degree of the cuspidal curve K , which by definition is given by the intersection number of K with a hyperplane section H .

Lemma 4.9. *If ℓ is a separable quasi-elliptic line of degree 3, then the degree of its cuspidal curve is at least 3 and at most 7.*

Proof. Writing $H = F + L$, one gets $k := K \cdot H = 3 + K \cdot L$. The cuspidal curve K and the line L are distinct because ℓ is separable. The curve K can meet L only in points of ramification; moreover, a local computation shows that if K is tangent to L , then ramification 3_4 occurs, and that higher order tangency cannot happen. We thus obtain the following bounds according to the ramification type of ℓ :

- 2_1^4 : $K \cdot L \leq 4$.
- $3_2 2_1$: $K \cdot L \leq 2$.
- 3_4 : $K \cdot L \leq 2$. □

Proposition 4.10. *If ℓ is a separable quasi-elliptic line of degree 3, then $v(\ell) \leq 21$.*

Proof. A fiber of type II^* can have local valency at most 2, because it contains only one simple components and three distinct lines would give rise to three distinct simple components. Hence, recalling equation (8),

$$\begin{aligned} v(\ell) &\leq 3iv + 3iv^* + 2ii^* \\ &= 3(10 - 3iv^* - 4ii^*) + 3iv^* + 2ii^* \\ &= 30 - 6iv^* - 10ii^*. \end{aligned}$$

In particular, if $ii^* > 0$, then $v(\ell) \leq 20$, so we can suppose that ℓ has no II^* -fibers. Similarly, we can suppose that ℓ has at most one fiber of type IV^* .

If ℓ has no IV^* -fiber, then it must have 10 fibers of type IV . Using the classification of Lemma 4.7, we list the possible configurations with $v(\ell) > 21$ (16 cases) in the following table.

case	iv^*	iv_3	iv_1	iv_0	valency
1	—	10	0	0	30
2	—	9	1	0	28
3	—	9	0	1	27
4	—	8	2	0	26
5	—	8	1	1	25
6	—	8	0	2	24
7	—	7	3	0	24
8	—	7	2	1	23
9	—	7	1	2	22
10	—	6	4	0	22
11	iv_3^*	7	0	0	24
12	iv_3^*	6	1	0	22
13	iv_{2a}^*	7	0	0	23
14	iv_{2b}^*	7	0	0	23
15	iv_{1a}^*	7	0	0	22
16	iv_{1t}^*	7	0	0	22

For each case, we consider the lattice generated by L , a general fiber F , the fiber components of the degenerate fibers and the cuspidal curve K (which

must be different from L , since ℓ is separable). All intersection numbers are univocally determined ($L \cdot F = 3$ because ℓ has degree 3), except for

$$K \cdot L = K \cdot (H - F) = k - 3,$$

but k can only take up the values $3, \dots, 7$ on account of Lemma 4.9. We check that this lattice has rank bigger than 22 in all cases, except for case 6 with $k = 3$ (i.e., $K \cdot L = 0$).

On the other hand, this case does not exist. In fact, suppose that ℓ is as in case 6 with $K \cdot L = 0$; in particular, ℓ has no ramified fibers with multiple components and, since $v(\ell) = 24$, ℓ is of the second kind. It follows that

- if ℓ has a point of ramification 2, then by Lemma 3.1, the ramified fiber must be a cusp, i.e., K intersects L so $K \cdot L > 0$;
- if ℓ has ramification 3_4 , then by Lemma 3.2 the ramified fiber must be either a cusp or the union of three distinct lines; in both cases, $K \cdot L > 0$. \square

Example 4.11. The following surface contains a separable quasi-elliptic line $\ell : x_0 = x_1 = 0$ of degree 3 with valency 21, thus attaining the bound of Proposition 4.10:

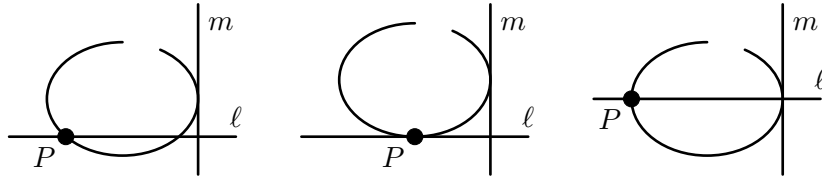
$$X : x_1^4 + x_0^2 x_2^2 - x_1^2 x_2^2 - x_1 x_2^3 + x_0 x_2^2 x_3 + x_0 x_3^3.$$

It contains only one singular point $[1 : 0 : 0 : 0]$ of type \mathbf{E}_6 .

4.2. Quasi-elliptic lines of lower degree.

Proposition 4.12. *If ℓ is a quasi-elliptic line of degree 2, then $v(\ell) \leq 14$.*

Proof. By Proposition 2.9, we can assume that ℓ is of the second kind. Let P be the singular point on ℓ and let F be a fiber of ℓ with $v_\ell(F) > 0$ and C its corresponding residual cubic. The cubic C is reducible, because it contains at least a line. Suppose that C splits into a line m and an irreducible conic (which must be tangent to each other because fibers of type \mathbf{I}_n are not admitted in a quasi-elliptic fibration). Since $v_\ell(F) > 0$, the line m meets ℓ in a smooth point; hence, the following three configurations may arise:



All three configurations are impossible for the following reasons:

- in the first configuration, the conic meets ℓ in a non-inflection point (the smooth surface point), by Lemma 2.6;
- the second configuration can be ruled out by an explicit parametrization (in a line of the second kind, either the point P is a ramification point, or the cubic passing twice through P is singular at P);
- the third configuration gives rise to a fiber of type \mathbf{III} , which is not a quasi-elliptic fiber.

Thus, C must split into three (not necessarily distinct) lines and at least one of them should pass through P . Since there can be at most 8 lines

through a singular point (Lemma 2.1), there can be at most 7 such reducible fibers, each of them contributing at most 2 to the valency of ℓ , whence $v(\ell) \leq 14$. \square

Example 4.13. The bound given by Proposition 4.12 is sharp. In fact, the following quartic surface contains a quasi-elliptic line $\ell : x_0 = x_1 = 0$ of degree 2 and valency 14:

$$X : x_0^4 + x_0^3x_1 + x_0x_1^3 + x_1x_2^3 + x_0x_1x_3^2 + x_1^2x_3^2 + x_0x_2x_3^2 = 0.$$

The quartic contains two singular points, $P = [0 : 0 : 0 : 1]$ of type \mathbf{A}_1 and $Q = [-1 : 1 : 1 : 0]$ of type \mathbf{E}_6 . The line ℓ has 7 fibers of type IV and one fiber of type IV* corresponding to the plane containing Q .

Lemma 4.14. *If ℓ is a quasi-elliptic line of degree 1, then $v(\ell) \leq 10$.*

Proof. The fibration induced by the line ℓ has at most 10 reducible fibers, each of which contributes at most 1 to its valency. \square

5. PROOF OF THEOREM 1.1

5.1. Triangle free case. In this section we employ the notation and the ideas of §2.4 and [19, §5].

Proposition 5.1. *Let Γ be the line graph of a triangle free K3 quartic surface. If Γ contains a parabolic subgraph D , then*

$$|\Gamma| \leq v(D) + 24$$

Proof. The subgraph Γ induces a genus 1 fibration, which can be elliptic or quasi-elliptic [8, §3, Theorem 1]. The vertices in $D \cup (\Gamma \setminus \text{span } D)$ are fiber components of this fibration. If the fibration is elliptic, there cannot be more than 24 components, on account of the Euler–Poincaré characteristic. If the fibration is quasi-elliptic, we obtain from formula (8) that

$$(10) \quad iv^* + ii^* \leq 3$$

A fiber of type IV can contain at most 2 lines, since there are no triangles. Hence, from (8) and (10) we deduce

$$\begin{aligned} |\Gamma| &\leq v(D) + 2iv + 7iv^* + 9ii^* \\ &= v(D) + 20 + iv^* + ii^* \\ &\leq v(D) + 23. \end{aligned} \quad \square$$

Lemma 5.2. *If ℓ is a line on a triangle free K3 quartic surface, then $v(\ell) \leq 12$.*

Proof. Thanks to Lemma 2.10, we can suppose that ℓ is quasi-elliptic. We can prove that

$$(11) \quad 2v_\ell(F) \leq e(F) - 2,$$

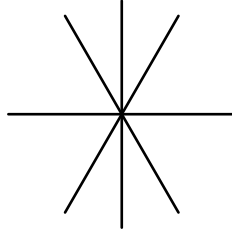
for any degenerate fiber F , which together with the Euler–Poincaré characteristic formula and equation (2) yields $v(\ell) \leq 10$. To see this, note that a reducible fiber has $e(F) \geq 4$, so (11) is obvious for a 1-fiber. On the other hand, a 3-fiber cannot be of type IV, in virtue of the triangle free hypothesis; hence, it must induce a fiber of type IV* or II*, for which $e(F) \geq 8$ and (11) holds. \square

Proposition 5.3. *A triangle free K3 quartic surface can contain at most 64 lines.*

Proof. One can adapt the proof in [19, Proposition 5.9] using Proposition 5.1 and Lemma 5.2. \square

5.2. Star case. We will suppose from now on that X has a triangle formed by the lines ℓ_1, ℓ_2, ℓ_3 , which are necessarily coplanar. The plane on which they lie must contain a fourth line ℓ_4 (which might coincide with one of the former). We will start our analysis with the following special configuration.

Definition 5.4. A *star* on a quartic surface X is the union of four distinct lines meeting in a smooth point.



Since the four lines in a star are necessarily coplanar, a star is the same as a configuration \mathcal{C}_0 (Figure 1). The lines have necessarily degree 3 because there are no singular points on the plane containing them.

We will be able to prove in Proposition 5.13 that if X contains a star and is not projectively equivalent to the Fermat surface, then $\Phi(X) \leq 58$, which is a sharp bound.

We will first need a series of lemmas. In all of them, we will parametrize the surface X as in (4) in such a way that the star is contained in the plane $x_0 = 0$ and the lines meet at $[0 : 0 : 0 : 1]$, i.e., setting the following coefficients equal to 0:

$$a_{0301}, a_{0211}, a_{0121}, a_{0202}, a_{0112}, a_{0103}.$$

If necessary, we will parametrize a second line in the star ℓ' as $x_0 = x_2 = 0$, by further assuming $a_{0400} = 0$.

Lemma 5.5. *If ℓ is a line of the first kind in a star, then $v(\ell) \leq 15$.*

Proof. It can be checked by an explicit computation that the resultant of ℓ has a root of order 6 at the center of the star; this implies that there are at most $18 - 6 = 12$ lines meeting ℓ not contained in the star. \square

Lemma 5.6. *If ℓ is a separable line of ramification $2_1 3_3$ contained in a star, then it is of the first kind.*

Proof. By a change of coordinates, we can assume that the point of ramification index 2 is $[0 : 0 : 1 : 0]$, and that ramification occurs at $x_1 = 0$. Imposing that ℓ is of the second kind leads to a contradiction (ℓ cannot be separable). \square

Lemma 5.7. *If three lines in a star are separable and at least two of them have ramification 3_4 , then the third one also has ramification 3_4 .*

Proof. Beside $\ell : x_0 = x_1 = 0$ and $\ell' : x_0 = x_2 = 0$, we can suppose without loss of generality that a third line is given by $\ell'' : x_0 = x_1 + x_2 = 0$, setting $a_{0220} = a_{0130} + a_{0310}$. The conditions for ℓ , ℓ' or ℓ'' to be of ramification 3_4 are $a_{1012} = 0$, $a_{1102} = 0$ and $a_{1012} = a_{1102}$, respectively. Clearly, two of them imply the third one. \square

Lemma 5.8. *If three lines in a star are separable, then at most two of them can be of the second kind.*

Proof. We parametrize ℓ , ℓ' and ℓ'' as in the previous Lemma. Imposing that all three of them are of the second kind leads to a contradiction (at least one of them must be inseparable). \square

Lemma 5.9. *Let ℓ and ℓ' be two lines in a star; if ℓ is a separable line of the second kind, and ℓ' is a line of the first kind of ramification 3_4 , then $v(\ell') \leq 12$.*

Proof. This can be checked again by an explicit computation of the resultant of ℓ' , which has now a root of order 9 at the center of the star. \square

Lemma 5.10. *Let ℓ and ℓ' be two lines in a star; if ℓ is a cuspidal line, and ℓ' is not cuspidal, then $v(\ell') \leq 12$.*

Proof. We parametrize $\ell : x_0 = x_1 = 0$ as in Corollary 4.3. By virtue of Lemma 4.2, we can suppose that $\ell' : x_0 = x_2 = 0$ is separable. An explicit computation shows that ℓ' cannot be of the second kind, and that its resultant has a root of order 9 in $x_2 = 0$. \square

Lemma 5.11. *Let ℓ , ℓ' and ℓ'' be three lines in a star; if ℓ and ℓ' are cuspidal, and ℓ'' is not cuspidal, then $v(\ell'') = 3$.*

Proof. We parametrize $\ell : x_0 = x_1 = 0$ and $\ell' : x_0 = x_2 = 0$ as in Corollary 4.3, i.e., we suppose that X is given by the family \mathcal{C} where the following coefficients are set to zero:

$$a_{0400}, a_{0301}; a_{1201}, a_{1300}; a_{2200}, a_{2101}, a_{1210}.$$

By a further rescaling we put $a_{0310} = 1$ and we consider $\ell'' : x_0 = x_1 - x_2 = 0$. The line ℓ'' is inseparable and we can compute its polynomial φ as in formula (9) in the proof of Lemma 4.2 (by parametrizing the pencil with $x_0 = s^3(x_1 - x_2)$), which turns out to be

$$\varphi(s) = a_{2110}s^8.$$

This means that ℓ'' has only one singular fiber in $s = 0$ (namely a fiber of type IV with the maximum possible index of wild ramification), unless $a_{2110} = 0$ and $\varphi \equiv 0$, in which case ℓ'' is cuspidal. \square

Lemma 5.12. *If ℓ is a cuspidal line which is not contained in at least two stars, then $v(\ell) \leq 6$.*

Proof. On account of Lemma 4.4, the number of stars in which ℓ is contained is exactly equal to the number of fibers of type IV in its fibration; moreover, $v_\ell(F) = 1$ if F is of type IV* or II*, yielding

$$v(\ell) = 3iv + iv^* + i^*.$$

Recalling formula (8), we deduce that if $iv < 2$ then $v(\ell) \leq 6$. \square

Proposition 5.13. *If X contains a star and is not projectively equivalent to the Fermat surface, then X contains at most 58 lines.*

Proof. Let $\ell_1, \ell_2, \ell_3, \ell_4$ be the lines contained in the star. We will always use the bound (1), which takes the form

$$\Phi(X) \leq 4 + \sum_{i=1}^4 (v(\ell_i) - 3) = \sum_{i=1}^4 v(\ell_i) - 8.$$

(1) Suppose first that all lines ℓ_i are not cuspidal.

- If $v(\ell_i) \leq 15$ for $i = 1, 2, 3, 4$, then

$$\Phi(X) \leq 4 \cdot 15 - 8 = 52.$$

- If $v(\ell_1) > 15$, then by Lemmas 5.5 and 4.2, ℓ_1 must be separable of the second kind; hence $v(\ell_1) \leq 21$; if $v(\ell_i) \leq 15$ for $i = 2, 3, 4$, then

$$\Phi(X) \leq (21 + 3 \cdot 15) - 8 = 58.$$

- If $v(\ell_1) > 15$ and $v(\ell_2) > 15$, then by the same token both ℓ_1 and ℓ_2 are separable lines of the second kind. On account of Lemma 5.6, they both have ramification 3_4 . We claim that both $v(\ell_3)$ and $v(\ell_4)$ are not greater than 12. In fact, if ℓ_3 is separable, then by Lemmas 5.7 and 5.8 it must be of the first kind and have ramification 3_4 , which in turn implies that $v(\ell_3) \leq 12$, because of Lemma 5.9; if ℓ_3 is inseparable, then $v(\ell) \leq 12$ by Lemma 4.2. The same applies to ℓ_4 . Hence, we conclude that

$$\Phi(X) \leq (2 \cdot 21 + 2 \cdot 12) - 8 = 58.$$

(2) Assume now that exactly one of the lines, say ℓ_1 , is cuspidal, so that $v(\ell_1) \leq 30$. On account of Lemma 5.10 we have

$$\Phi(X) \leq (30 + 3 \cdot 12) - 8 = 58.$$

(3) Suppose then that both ℓ_1 and ℓ_2 are cuspidal. If ℓ_3 and ℓ_4 are not cuspidal, then by Lemma 5.11

$$\Phi(X) \leq (2 \cdot 30 + 2 \cdot 3) - 8 = 58.$$

(4) Finally, suppose that ℓ_1, ℓ_2 and ℓ_3 are cuspidal.

- By a local computation it can be seen that ℓ_4 is also necessarily cuspidal.
- Thanks to the bound of Lemma 5.12, we can suppose that at least two lines, say ℓ_1 and ℓ_2 , are part of another star.
- Pick two lines ℓ'_1 and ℓ'_2 , each of them in another star containing ℓ_1 respectively ℓ_2 , which intersect each other (necessarily in a smooth point).
- Perform a change of coordinates so that ℓ_1, ℓ_2, ℓ'_1 and ℓ'_2 are given respectively by $x_0 = x_1 = 0$, $x_0 = x_2 = 0$, $x_1 = x_3 = 0$ and $x_2 = x_3 = 0$.
- Impose that ℓ_1, ℓ_2 and ℓ_3 are cuspidal lines: the resulting surface is projectively equivalent to Fermat surface. \square

5.3. Triangle case. In this section we study the case in which X admits a triangle. The three lines forming the triangle need to be coplanar, and we will denote by Π the plane on which they lie. Obviously, the plane Π intersects X also in a fourth line, which might coincide with one of the first three. We first consider this degenerate case.

Proposition 5.14. *If X admits a completely reducible plane Π with a triangle and a multiple component, then X contains at most 60 lines.*

Proof. By Lemma 2.11, X admits configuration \mathcal{D}_0 or \mathcal{E}_0 . In order to bound $\Phi(X)$, we use as usual formula (1).

Let ℓ_0 be the double line in the plane Π containing one of the two configurations, and let ℓ_1 and ℓ_2 be the two simple lines. Lines meeting ℓ_0 different from ℓ_1 and ℓ_2 must pass through the singular points; hence, by Lemma 2.1 there can be at most $3 \cdot (8 - 1) = 21$ of them.

Note that ℓ_1 and ℓ_2 cannot be cuspidal because of Corollary 4.4.

In the fibrations induced by ℓ_1 and ℓ_2 the plane Π corresponds to a fiber with a multiple component, hence with Euler–Poincaré characteristic at least 6; therefore, if ℓ_1 and ℓ_2 are both elliptic, there can be at most 18 more lines meeting them, yielding

$$\Phi(X) \leq 3 \cdot (8 - 1) + (18 + 18) + 3 = 60.$$

Suppose that ℓ_1 is quasi-elliptic. The plane Π corresponds to a fiber of type IV^* or II^* ; hence, there can be only one singular point on ℓ_0 : in fact, by inspection of the Dynkin diagrams, a component of multiplicity 2 in these fiber types meets at most 2 other components (and one of them is the strict transform of ℓ_2). The lines ℓ_1 and ℓ_2 not being cuspidal, we know that they have valency at most 21. It follows that

$$\Phi(X) \leq (8 - 1) + 2 \cdot (21 - 2) + 3 = 48. \quad \square$$

Lemma 5.15. *Let ℓ and ℓ' be two lines of degree 3 in configuration \mathcal{A}_0 or \mathcal{A}_1 . If $v(\ell) > 18$, then $v(\ell') \leq 18$.*

Proof. Let Π be the plane containing ℓ and ℓ' . Both lines are separable, since otherwise the respective residual cubics would intersect them in one point. We suppose that also $v(\ell') > 18$ and look for a contradiction.

Since both lines have valency greater than 18, they must be lines of the second kind with ramification $(3(3), 2)$ or 3_4 . In particular, they must have a point of ramification 3 (let us call it $P \in \ell$ and $P' \in \ell'$), which does not lie on Π . Up to change of coordinates, we can assume the following:

- Π is the plane $x_0 = 0$;
- ℓ and ℓ' are given respectively by $x_0 = x_1 = 0$ and $x_0 = x_2 = 0$;
- P is given by $[0 : 0 : 1 : 0]$ and P' by $[0 : 1 : 0 : 0]$;
- ramification in P (resp. P') occurs in $x_1 = 0$ (resp. $x_2 = 0$).

This amounts to setting the following coefficients equal to 0:

$$a_{0400}, a_{0301}, a_{0202}, a_{0103}; a_{1030}, a_{1021}, a_{1012}; a_{1300}, a_{1201}, a_{1102}.$$

Furthermore, $a_{0112} \neq 0$, since the two residual lines in Π do not contain $[0 : 0 : 0 : 1]$, the intersection point of ℓ and ℓ' ; we set $a_{0112} = 1$ after rescaling one variable.

Two necessary condition for ℓ and ℓ' to be lines of the second kind are

$$a_{0310} = a_{0211}^2 \quad \text{and} \quad a_{0130} = a_{0121}^2.$$

This means that the residual conic in $\Pi : x_0 = 0$ is given explicitly by

$$(12) \quad a_{0211}^2 x_1^2 + a_{0121}^2 x_2^2 + a_{0220} x_1 x_2 + a_{0211} x_1 x_3 + a_{0121} x_2 x_3 + x_3^2 = 0.$$

This conic splits into two lines by hypothesis; hence, it has a singular point. Computing the derivatives, one finds that the following condition must be satisfied:

$$a_{0220} = -a_{0121} a_{0211}.$$

Substituting into (12), one finds that the conic degenerates to a double line:

$$(a_{0211} x_1 + a_{0121} x_2 - x_3)^2 = 0;$$

thus, we have neither configuration \mathcal{A}_0 nor \mathcal{A}_1 . \square

Proposition 5.16. *If X admits a triangle but not a star, then X contains at most 67 lines.*

Proof. The proof is a case-by-case analysis on the configurations that are given by Lemma 2.11, except configurations \mathcal{C} (a star, treated in Proposition 5.13), \mathcal{D}_0 and \mathcal{E}_0 (treated in Proposition 5.14).

Beside the fact that there are at most 8 lines through a singular point and the bounds on the valency of §3 and §4, one should observe that in configurations of type \mathcal{B} , the three lines meeting at the same (smooth) point must be of the first kind by Lemma 3.1. For configurations \mathcal{A}_0 and \mathcal{A}_1 , one uses Lemma 5.15.

For instance, let us prove the proposition for configuration \mathcal{A}_1 . Let ℓ_1 and ℓ_2 be the lines through the singular point, and ℓ_3 and ℓ_4 the other two lines. We know that $v(\ell_i) \leq 14$, $i = 1, 2$, whereas Lemma 5.15 applies to ℓ_3 and ℓ_4 , yielding

$$v(\ell_3) + v(\ell_4) \leq 18 + 21.$$

It follows from (1) that

$$\Phi(X) \leq (8 - 2) + 2 \cdot (14 - 2) + (18 - 3 + 21 - 3) + 4 = 67.$$

We leave the remaining cases to the reader. \square

Proof of Theorem 1.1. We have treated the case of triangle free surfaces in Proposition 5.3, the star case in Proposition 5.13 and the star free triangle case in Proposition 5.16, so the proof is now complete. \square

6. EXAMPLES

In this last section, we present examples of K3 quartic surfaces with many lines. In particular, we provide explicit equations for three 1-dimensional families of surfaces with 58 lines. Most of the examples – including the first two families (Examples 6.1 and 6.2) – were found during the proof of the theorem, especially of Proposition 5.13. Note that the first two families had already been discovered independently by A. Degtyarev, who was also aware of the existence of a third configuration with 58 lines. We found the third family (Example 6.3) after we were informed of his work; as far as we know, this family is new.

Example 6.1. A general member of the 1-dimensional family defined by

$$x_1^3x_2 - x_1x_2^3 + x_0^3x_3 - x_0x_3^3 = ax_0^2x_1x_2$$

is smooth and contains 58 lines.

More precisely, for $a = 0$ we obtain a surface which is projectively equivalent over \mathbb{F}_9 to the Fermat surface and thus contains 112 lines.

If $a \neq 0, \infty$, the surface contains a star (in $x_0 = 0$) formed by two cuspidal lines (of valency 30) and two elliptic lines with no other singular fibers than the star itself (hence, of valency 3). The remaining 54 lines are of type $(p, q) = (1, 9)$. The surface contains exactly 19 stars.

For $a = \infty$ we obtain the union of three planes.

Example 6.2. A general member of the 1-dimensional family defined by

$$x_1^3x_2 - x_1x_2^3 + x_0^3x_3 - x_0x_3^3 = ax_0x_1(ax_0x_2 + ax_1x_3 + x_1x_2 + x_0x_3)$$

is smooth and contains exactly 58 lines.

More precisely, as long as $a \neq 0, 1, -1, \infty$, the surface contains one cuspidal line (given by $x_0 = x_1 = 0$) which intersects 12 lines of type $(4, 0)$, and 18 lines of type $(1, 9)$; the remaining 27 lines are of type $(4, 6)$ (for instance, $x_2 = x_3 = 0$). The surface contains exactly 10 stars.

For $a = 0$ we find again a model of the Fermat surface, whereas for $a = \pm 1$ the surface contains 20 lines and a triple point. For $a = \infty$ we obtain the union of two planes and a quadric surface.

All surfaces of the family are endowed with the symmetries $[x_0 : x_1 : x_2 : x_3] \rightsquigarrow [x_1 : x_0 : x_3 : x_2]$ and $[x_0 : x_1 : x_2 : x_3] \rightsquigarrow [x_0 : x_1 : -x_2 : -x_3]$.

Example 6.3. A general member of the 1-dimensional family defined by

$$\begin{aligned} & (a^3 + a^2 + a + 1)(x_1^3x_2 + x_1x_2^3 - x_0^3x_3 - x_0x_3^3) = \\ & (a - 1)(x_0^2x_1x_2 - x_0^2x_3^2 + x_1x_2x_3^2) + (a + 1)(x_1^2x_2^2 - x_0x_1^2x_3 - x_0x_2^2x_3) \\ & + (a^2 - 1)(x_1x_2 + x_0x_3)(x_0 + x_3)(x_1 + x_2) - (a^2 + 1)x_0x_1x_2x_3 \end{aligned}$$

is smooth and contains exactly 58 lines.

More precisely, if $a \neq 0, 1, -1, \infty$ and $a^2 \neq -1$, then the surface contains exactly one star in the plane

$$(13) \quad x_0 + x_3 = x_1 + x_2.$$

The star is formed by two lines of type $(7, 0)$ and two lines of type $(1, 9)$, whose equations can be explicitly written after a change of parameter $a = d/(d^2 + 1)$. Each line of type $(7, 0)$ meets 18 lines of type $(3, 6)$, and each line of type $(1, 9)$ meets 9 lines of type $(4, 6)$. All lines are elliptic.

If $a = 0, 1$ or -1 the surface is the union of a double plane and a quadric surface. If $a = \infty$ the surface is projectively equivalent to the Fermat surface.

If $a^2 = -1$, then the surface contains 9 points of type \mathbf{A}_1 and 40 lines. The star in the plane (13) is formed by two elliptic lines of type $(4, 0)$ and two quasi-elliptic lines of type $(1, 9)$. Each line of type $(4, 0)$ intersects 9 lines of singularity 2 and valency 6, while each line of type $(1, 9)$ intersects 9 lines of singularity 1 and valency 9.

All surfaces of the family are endowed with the symmetries $[x_0 : x_1 : x_2 : x_3] \rightsquigarrow [x_1 : x_0 : x_2 : x_3]$ and $[x_0 : x_1 : x_2 : x_3] \rightsquigarrow [x_0 : x_1 : x_3 : x_2]$.

Example 6.4. The surface defined by

$$\begin{aligned} & x_0^3x_1 + x_0^2x_1^2 + x_0x_1^2x_2 + x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3 + x_0^3x_3 \\ & - x_0^2x_1x_3 + x_0x_1^2x_3 + x_0^2x_2x_3 + x_0x_1x_2x_3 + x_0x_2^2x_3 + x_0x_3^3 = 0 \end{aligned}$$

contains one singular point of type \mathbf{E}_7 and 39 lines.

Example 6.5. The reduction modulo 3 of Shimada–Shioda’s surface X_{56} (see [16]) can be written

$$\Psi(x_0, x_1, x_2, x_3) = \Psi(-x_1, x_0, -x_3, x_2)$$

where

$$\Psi(w, x, y, z) = wz(w^2 + wx + x^2 + y^2 + yz + z^2)$$

It contains 8 singular points of type \mathbf{A}_1 and 48 lines. So far, this is the example known to us with highest number of lines and at least one singular point.

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